

Section A: Pure Mathematics

1 Consider the equations

$$\begin{aligned}ax - y - z &= 3, \\2ax - y - 3z &= 7, \\3ax - y - 5z &= b,\end{aligned}$$

where a and b are given constants.

- (i) In the case $a = 0$, show that the equations have a solution if and only if $b = 11$.
- (ii) In the case $a \neq 0$ and $b = 11$ show that the equations have a solution with $z = \lambda$ for any given number λ .
- (iii) In the case $a = 2$ and $b = 11$ find the solution for which $x^2 + y^2 + z^2$ is least.
- (iv) Find a value for a for which there is a solution such that $x > 10^6$ and $y^2 + z^2 < 1$.

2 Write down a value of θ in the interval $\pi/4 < \theta < \pi/2$ that satisfies the equation

$$4 \cos \theta + 2\sqrt{3} \sin \theta = 5.$$

Hence, or otherwise, show that

$$\pi = 3 \arccos(5/\sqrt{28}) + 3 \arctan(\sqrt{3}/2).$$

Show that

$$\pi = 4 \arcsin(7\sqrt{2}/10) - 4 \arctan(3/4).$$

3 Prove that the cube root of any irrational number is an irrational number.

Let $u_n = 5^{1/(3^n)}$. Given that $\sqrt[3]{5}$ is an irrational number, prove by induction that u_n is an irrational number for every positive integer n .

Hence, or otherwise, give an example of an infinite sequence of irrational numbers which converges to a given integer m .

[An irrational number is a number that cannot be expressed as the ratio of two integers.]

- 4 The line $y = d$, where $d > 0$, intersects the circle $x^2 + y^2 = R^2$ at G and H . Show that the area of the minor segment GH is equal to

$$R^2 \arccos\left(\frac{d}{R}\right) - d\sqrt{R^2 - d^2}. \quad (*)$$

In the following cases, the given line intersects the given circle. Determine how, in each case, the expression (*) should be modified to give the area of the minor segment.

- (i) Line: $y = c$; circle: $(x - a)^2 + (y - b)^2 = R^2$.
- (ii) Line: $y = mx + c$; circle: $x^2 + y^2 = R^2$.
- (iii) Line: $y = mx + c$; circle: $(x - a)^2 + (y - b)^2 = R^2$.

- 5 The position vectors of the points A , B and P with respect to an origin O are $a\mathbf{i}$, $b\mathbf{j}$ and $l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$, respectively, where a , b , and n are all non-zero. The points E , F , G and H are the midpoints of OA , BP , OB and AP , respectively. Show that the lines EF and GH intersect.

Let D be the point with position vector $d\mathbf{k}$, where d is non-zero, and let S be the point of intersection of EF and GH . The point T is such that the mid-point of DT is S . Find the position vector of T and hence find d in terms of n if T lies in the plane OAB .

- 6 The function f is defined by

$$f(x) = |x - 1|,$$

where the domain is \mathbf{R} , the set of all real numbers. The function $g_n = f^n$, with domain \mathbf{R} , so for example $g_3(x) = f(f(f(x)))$. In separate diagrams, sketch graphs of g_1 , g_2 , g_3 and g_4 .

The function h is defined by

$$h(x) = \left| \sin \frac{\pi x}{2} \right|,$$

where the domain is \mathbf{R} . Show that if n is even,

$$\int_0^n (h(x) - g_n(x)) \, dx = \frac{2n}{\pi} - \frac{n}{2}.$$

- 7 Show that, if $n > 0$, then

$$\int_{e^{1/n}}^{\infty} \frac{\ln x}{x^{n+1}} dx = \frac{2}{n^2 e}.$$

You may assume that $\frac{\ln x}{x} \rightarrow 0$ as $x \rightarrow \infty$.

Explain why, if $1 < a < b$, then

$$\int_b^{\infty} \frac{\ln x}{x^{n+1}} dx < \int_a^{\infty} \frac{\ln x}{x^{n+1}} dx.$$

Deduce that

$$\sum_{n=1}^N \frac{1}{n^2} < \frac{e}{2} \int_{e^{1/N}}^{\infty} \left(\frac{1 - x^{-N}}{x^2 - x} \right) \ln x dx,$$

where N is any integer greater than 1.

- 8 It is given that y satisfies

$$\frac{dy}{dt} + k \left(\frac{t^2 - 3t + 2}{t + 1} \right) y = 0,$$

where k is a constant, and $y = A$ when $t = 0$, where A is a positive constant. Find y in terms of t , k and A .

Show that y has two stationary values whose ratio is $(3/2)^{6k} e^{-5k/2}$.

Describe the behaviour of y as $t \rightarrow +\infty$ for the case where $k > 0$ and for the case where $k < 0$.

In separate diagrams, sketch the graph of y for $t > 0$ for each of these cases.

Section B: Mechanics

- 9 AB is a uniform rod of weight W . The point C on AB is such that $AC > CB$. The rod is in contact with a rough horizontal floor at A and with a cylinder at C . The cylinder is fixed to the floor with its axis horizontal. The rod makes an angle α with the horizontal and lies in a vertical plane perpendicular to the axis of the cylinder. The coefficient of friction between the rod and the floor is $\tan \lambda_1$ and the coefficient of friction between the rod and the cylinder is $\tan \lambda_2$.

Show that if friction is limiting both at A and at C , and $\alpha \neq \lambda_2 - \lambda_1$, then the frictional force acting on the rod at A has magnitude

$$\frac{W \sin \lambda_1 \sin(\alpha - \lambda_2)}{\sin(\alpha + \lambda_1 - \lambda_2)}.$$

- 10 A bead B of mass m can slide along a rough horizontal wire. A light inextensible string of length 2ℓ has one end attached to a fixed point A of the wire and the other to B . A particle P of mass $3m$ is attached to the mid-point of the string and B is held at a distance ℓ from A . The bead is released from rest.

Let a_1 and a_2 be the magnitudes of the horizontal and vertical components of the initial acceleration of P . Show by considering the motion of P relative to A , or otherwise, that $a_1 = \sqrt{3}a_2$. Show also that the magnitude of the initial acceleration of B is $2a_1$.

Given that the frictional force opposing the motion of B is equal to $(\sqrt{3}/6)R$, where R is the normal reaction between B and the wire, show that the magnitude of the initial acceleration of P is $g/18$.

- 11 A particle P_1 is projected with speed V at an angle of elevation α ($> 45^\circ$), from a point in a horizontal plane. Find T_1 , the flight time of P_1 , in terms of α, V and g . Show that the time after projection at which the direction of motion of P_1 first makes an angle of 45° with the horizontal is $\frac{1}{2}(1 - \cot \alpha)T_1$.

A particle P_2 is projected under the same conditions. When the direction of the motion of P_2 first makes an angle of 45° with the horizontal, the speed of P_2 is instantaneously doubled. If T_2 is the total flight time of P_2 , show that

$$\frac{2T_2}{T_1} = 1 + \cot \alpha + \sqrt{1 + 3 \cot^2 \alpha}.$$

SECTION C: Probability and Statistics

- 12** The life of a certain species of elementary particles can be described as follows. Each particle has a life time of T seconds, after which it disintegrates into X particles of the same species, where X is a random variable with binomial distribution $B(2, p)$. A population of these particles starts with the creation of a single such particle at $t = 0$. Let X_n be the number of particles in existence in the time interval $nT < t < (n + 1)T$, where $n = 1, 2, \dots$

Show that $P(X_1 = 2 \text{ and } X_2 = 2) = 6p^4q^2$, where $q = 1 - p$. Find the possible values of p if it is known that $P(X_1 = 2 | X_2 = 2) = 9/25$.

Explain briefly why $E(X_n) = 2pE(X_{n-1})$ and hence determine $E(X_n)$ in terms of p . Show that for one of the values of p found above $\lim_{n \rightarrow \infty} E(X_n) = 0$ and that for the other $\lim_{n \rightarrow \infty} E(X_n) = +\infty$.

- 13** The random variable X takes the values $k = 1, 2, 3, \dots$, and has probability distribution

$$P(X = k) = A \frac{\lambda^k e^{-\lambda}}{k!},$$

where λ is a positive constant. Show that $A = (1 - e^{-\lambda})^{-1}$. Find the mean μ in terms of λ and show that

$$\text{Var}(X) = \mu(1 - \mu + \lambda).$$

Deduce that $\lambda < \mu < 1 + \lambda$.

Use a normal approximation to find the value of $P(X = \lambda)$ in the case where $\lambda = 100$, giving your answer to 2 decimal places.

- 14** The probability of throwing a 6 with a biased die is p . It is known that p is equal to one or other of the numbers A and B where $0 < A < B < 1$. Accordingly the following statistical test of the hypothesis $H_0 : p = B$ against the alternative hypothesis $H_1 : p = A$ is performed.

The die is thrown repeatedly until a 6 is obtained. Then if X is the total number of throws, H_0 is accepted if $X \leq M$, where M is a given positive integer; otherwise H_1 is accepted. Let α be the probability that H_1 is accepted if H_0 is true, and let β be the probability that H_0 is accepted if H_1 is true.

Show that $\beta = 1 - \alpha^K$, where K is independent of M and is to be determined in terms of A and B . Sketch the graph of β against α .